

# Optimal risk allocation in a market with non-convex preferences

Hirbod Assa <sup>\*</sup>  
University of Liverpool

## Abstract

The aims of this study are twofold. First, we consider an optimal risk allocation problem with non-convex preferences. By establishing an infimal representation for distortion risk measures, we give some necessary and sufficient conditions for the existence of optimal and asymptotic optimal allocations. We will show that, similar to a market with convex preferences, in a non-convex framework with distortion risk measures the boundedness of the optimal risk allocation problem depends only on the preferences. Second, we consider the same optimal allocation problem by adding a further assumption that allocations are co-monotone. We characterize the co-monotone optimal risk allocations within which we prove the “marginal risk allocations” take only the values zero or one. Remarkably, we can separate the role of the market preferences and the total risk in our representation.

## 1 Introduction

There is considerable interest in the problem of optimal risk allocation, as it is at the heart of many financial and insurance applications. Optimal risk sharing, optimal capital allocation, theory of market equilibrium, optimal reinsurance design and optimal risk exchange are only a few examples. This problem dates back to the 50s and 60s when Allais (1953), Arrow (1964), Sharpe (1964), Borch (1960), Mossin (1966) and many others studied the optimal risk allocations for different

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<sup>\*</sup>Mailing address: Institute for Financial and Actuarial Mathematics. University of Liverpool. UK. Email: [assa@liverpool.ac.uk](mailto:assa@liverpool.ac.uk)

economic problems. Thereafter, researchers started to elaborate further on the aspects of this problem for a variety of assumptions. By development of risk measures and their applications in finance and insurance, the problem of optimal risk allocation has been revisited by using coherent risk measures of Artzner et al. (1999), convex risk measures of Föllmer and Schied (2002) and deviation measures of risk of Rockafellar et al. (2006). The first attempt to study the problem in a setting with coherent risk measures was Heath and Ku (2004), where the authors established a necessary and sufficient condition for the existence of a Pareto optimal allocation. Barrieu and El Karoui (2004) considered a risk sharing problem in a dynamic setup, whereas Jouini et al. (2008) considered a static framework with law-invariant convex risk measures. Filipović and Kupper (2008a) looked at the optimal risk allocation problem from a pricing point of view, while Filipović and Kupper (2008b) considered it for optimal capital allocations. Acciaio (2007) studied a sharing pooled risk problem with non-necessarily monotone monetary utilities. While there is extensive research on the problem of optimal risk allocation with convex preferences, studies using non-convex framework have been relatively scarce, whereas in many applications preferences are not convex, and the results of the existing settings cannot be applied to them. This is mainly due to the lack of appropriate mathematical techniques to study models with the non-convex preferences.

In this paper, by establishing an infimal representation for distortion risk measures, we find a new way to study the optimal risk allocation problem with non-convex preferences. We prove that the boundedness of the optimal risk allocation problem is independent of the total risk and only depends on the market preferences. The approach we have chosen is a finance oriented approach which gives rise to the definition of generalized stochastic discount factors for non-convex preferences (see Remark 2 below). Our results generalize results of Jouini et al. (2008), Filipović and Kupper (2008a) and Filipović and Kupper (2008b) towards a new direction by using non-convex risk measures. This constitutes the first part of the paper. In the second part, with an extra assumption that the risk allocations are co-monotone, we characterize the optimal risk allocations in the same market. This assumption can be interpreted as mutualization of risks, which is closely related to the moral hazard risk<sup>1</sup>. Interestingly, we see that the optimal risk allocations in a setting with distortion risk measures are in a perfect accordance with this assumption. It is shown in Filipović and Svindland (2008) that the solutions to a general market risk allocation problem with convex distortion risk measures are co-monotone, and therefore, rule out the risk of moral hazard. However, we will see within an example that this no

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<sup>1</sup> In order to avoid the moral hazard risk, allocations have to be increasing in terms of market risks.

longer holds true when agents use non-convex distortion risk measures. That is why we have to assume that the allocations are co-monotone. In order to characterize the co-monotone optimal solutions, we introduce the “marginal risk allocations”. A marginal risk allocation is the marginal rate of changes in the value of a contract when we marginally change the value of the total risk. It is shown that, in a market with co-monotone risk allocations, the marginal risk allocations take only the values zero or one. This way, we can remarkably separate the role of the market preferences and the total risk in the optimal risk allocations. Our results find a new characterization of the optimal allocations in Chateauneuf et al. (2000) enabling us for more precise interpretation of the optimal allocations and also finding further applications in other fields such as the optimal re-insurance design. This paper generalize the literature of optimal re-insurance design in two directions. First, we use a larger family of (non-convex) risk measures and premiums and second, we increase the number of players from two to  $n$  (e.g. see, Cai et al. (2008), Cheung (2010), Chi (2012b), Chi (2012a), Chi and Tan (2013), Cheung et al. (2014) and Assa (2015)).

The rest of the paper is organized as follows: in Section 2 we introduce the needed notions and notations, and recall some facts from convex analysis. In Section 3, first, we set up the main problem, second, we discuss some necessary and sufficient conditions for the existence of general solutions, and, third, we characterize the co-monotone optimal solutions.

## 2 Preliminaries and Notations

Throughout the paper, we will fix a probability space  $(\Omega, \mathcal{F}, P)$ , where  $\mathcal{F}$  is a  $\sigma$ -algebra and  $P$  is a probability measure on  $\mathcal{F}$ . Let  $p, q \in [1, \infty]$  be two numbers such that  $1/p + 1/q = 1$ . For  $p \neq \infty$ ,  $L^p$  denotes the space of real-valued random variables  $X$  on  $\Omega$  such that  $E(|X|^p) < \infty$ , where  $E$  represents the mathematical expectation. Recall that according to the Riesz Representation Theorem,  $L^q$  is the dual space of  $L^p$  when  $p \neq \infty$ . We endow the space  $L^p$  with two topologies, first the norm topology induced by  $\|X\|_p = E(|X|^p)^{\frac{1}{p}}$ , and second the weak topology, induced by  $L^q$  i.e. the coarsest topology in which all members of  $L^q$  are continuous. As usual the latter topology is denoted by  $\sigma(L^p, L^q)$ .

In this paper we consider that  $L^p$  represents the space of all loss variables<sup>2</sup>. We only have two periods of time 0 and  $T$ , representing the beginning of the year when a contract is written, and the end the year when liabilities are settled, respectively.

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<sup>2</sup>Unlike finance literature which consider profit variable, we found the loss variable more convenient to deal with.

Every random variable represents losses at time  $T$ . Whenever we talk about risk or premium we mean the present value of the loss and the premium at time  $T = 0$ .

## 2.1 Distortion Risk Measures

Let  $\Phi : [0, 1] \rightarrow [0, 1]$  be a non-decreasing and càdlàg function such that  $\Phi(0) = 1 - \Phi(1) = 0$ .  $\Phi$  can introduce a measure on  $[0, 1]$  whose values on the intervals are given as  $m_\Phi[a, b] = \Phi(b) - \Phi(a)$  and  $m_\Phi(b) = 1 - \lim_{a \uparrow 1} \Phi(a)$ . Introduce the set  $\mathcal{D}_\Phi$  as follows

$$\mathcal{D}_\Phi = \left\{ X \in L^0 \mid \int_0^1 \text{VaR}_t(X) d\Phi(t) \in \mathbb{R} \right\}, \quad (1)$$

where the integral above is the Lebesgue integral and

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid P(X > x) \leq 1 - \alpha\}, \alpha \in [0, 1].$$

**Definition 1.** A distortion risk measure  $\varrho_\Phi$  (or simply  $\varrho$ ) is a mapping from  $\mathcal{D}_\Phi$  to  $\mathbb{R}$  defined as

$$\varrho_\Phi(X) = \int_0^1 \text{VaR}_t(X) d\Phi(t), \quad (2)$$

If we let  $g(x) := 1 - \Phi(1 - x)$  one can see that

$$\varrho_\Phi(X) = \int_{-\infty}^0 (g(S_X(t)) - 1) dt + \int_0^\infty g(S_X(t)) dt, \quad (3)$$

where  $S_X = 1 - F_X$  is the survival function associated with  $X$ . Note that we can associate  $\varrho$  with  $\Phi$  by using the notation  $\Phi_\varrho$ . This is a Choquet integral representation of the risk measure. In the literature,  $g$  is known as the distortion function. A popular example is Value at Risk (VaR), whose distortion function is given by  $g(t) = 1_{[1-\alpha, 1]}(t)$  for a confidence level  $1 - \alpha$ . It can also explicitly be given as

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid P(X > x) \leq 1 - \alpha\}.$$

Another example of a distortion risk measure is Conditional Value at Risk (CVaR), when  $\Phi(t) = \frac{t-\alpha}{1-\alpha} 1_{[\alpha, 1]}(t)$  and can be represented in terms of VaR

$$\text{CVaR}_\alpha(x) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_t(X) dt. \quad (4)$$

The family of spectral risk measures which was introduced first in Acerbi (2002), is a distortion risk when  $\Phi$  is convex.

*Remark 1.* One can readily see that  $\varrho_\Phi$  is law invariant, i.e., if  $X$  and  $X'$  are identically distributed, then we have  $\varrho_\Phi(X) = \varrho_\Phi(X')$ . Indeed, it can be shown that all law-invariant co-monotone additive coherent risk measures can be represented as (2); see Kusuoka (2001). A risk measure in the form (2) is important from different perspectives. First of all, it makes a link between the risk measure theory and the behavioral finance as the form (2) is a particular form of distortion utility. Second, (2) contains a family of risk measures which are statistically robust. In Cont et al. (2010) it is shown that a risk measure  $\varrho(x) = \int_0^1 \text{VaR}_t(x) d\Phi(t)$  is robust if and only if the support of  $\varphi = \frac{d\Phi(t)}{dt}$ <sup>3</sup> is away from zero or one. For example Value at Risk is a risk measure with this property. Distortion utilities have become increasingly important in the literature of decision making since they take into account some known behavioral paradoxes such as the Allais paradox under risk and the Ellsberg paradox under uncertainty. Schmeidler (1989) (under uncertainty) and Quiggin (1982) and Yaari (1984), Yaari (1986) (under risk) show by assuming co-monotone independence, preferences are according to utilities which admit a distortion integral representation. It is worth mentioning that, distortion integrals have become very popular in the literature of insurance because they are the natural extensions of important insurance risk premiums such as Proportional Hazards Premium Principle, Wang's Premium Principle and Net Premium Principle (see Wang et al. (1997) and Young (2006)).

Finally, we have the definition of a coherent risk measure

**Definition 2.** A coherent risk measure  $\varrho$  is a lower semi-continuous<sup>4</sup>(see below for definition of lower semi-continuous) mapping from  $L^p$  to  $\mathbb{R} \cup \{+\infty\}$  such that

1.  $\varrho(\lambda X) = \lambda \varrho(X)$ , for all  $\lambda > 0$  and  $X \in L^p$ ;
2.  $\varrho(X + c) = \varrho(X) + c$ , for all  $X \in L^p$  and  $c \in \mathbb{R}$ ;
3.  $\varrho(X) \leq \varrho(Y)$ , for all  $X, Y \in L^p$  and  $X \leq Y$ ;
4.  $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$ ,  $\forall X, Y \in L^p$ ;

As one can see, a coherent risk measure is positive homogeneous. As we will see in the next section, there is a closed and convex subset  $\Delta_\varrho \subseteq L^q$ , such that  $\varrho(X) = \sup_{Y \in \Delta_\varrho} E(YX)$ . One can show that for any  $Y \in \Delta_\varrho$ , we have  $E(Y) = 1$  and  $Y \geq 0$ .

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<sup>3</sup> $\varphi$  is a general derivative of  $\Phi$ .

<sup>4</sup>The risk measure in general does not need to be lower semi-continuous in  $L^\infty$ , however we add it to be consistent with  $L^p, p \neq \infty$ .

## 2.2 Some Facts from Convex Analysis

Here we recall some relevant discussions from the convex analysis. Recalling from the convex analysis, for any convex function  $\phi$ , the domain of  $\phi$  denoted by  $\text{dom}(\phi)$  is equal to  $\{X \in L^p | \phi(X) < \infty\}$ , and the dual of  $\phi$ , denoted by  $\phi^*$ , is defined as  $\phi^*(Y) = \sup_{X \in L^p} E(XY) - \phi(X)$ . A convex function is called lower-semi-continuous iff  $\phi = \phi^{**}$ . In this paper, we assume all convex functions are lower semi continuous. For a convex set  $C \subseteq L^p$ , the indicator function of  $C$  is denoted by  $\chi_C$  and is equal to 0 if  $X \in C$ , and  $+\infty$ , otherwise. One can incorporate any type of convex restriction by using an appropriate indicator function. Let  $C$  be a closed and convex set representing a convex restriction on  $\phi$ . By introducing  $\phi^C = \phi + \chi_C$  we incorporate the restriction  $C$ . Note that  $\phi^C$  is a convex function.

For any positive homogeneous convex function  $\phi$  let

$$\Delta_\phi = \{Y \in L^q | E(YX) \leq \phi(X), \forall X \in L^p\}.$$

It is easy to see that  $\phi^* = \chi_{\Delta_\phi}$ . Therefore, any positive homogeneous function  $\phi$  can be represented as  $\phi(X) = \sup_{Y \in \Delta_\phi} E(YX)$ . By using this and that  $\phi = \phi^{**}$ , one

can easily see that for any convex set  $C$ ,  $\chi_C^*(Y) = \sup_{X \in C} E(YX)$ . As a result, if  $C$  is a convex cone and  $\phi$  is a positive homogeneous convex function then  $\phi^C(X) = \sup_{Y \in \Delta_\phi + C^\perp} E(YX)$ , where  $C^\perp = \{Y \in L^q; E(YX) \leq 0, \forall X \in C\}$  (or  $\Delta_{\phi^C} = \Delta_\phi + C^\perp$ ).

A particular interesting example is  $C = L_+^p$  when  $C^\perp = L_-^q$ .

For a set of convex functions  $\phi_1, \dots, \phi_n$  their infimal convolution is defined as

$$\phi_1 \square \dots \square \phi_n(X) = \inf_{X_1 + \dots + X_n = X} \phi_1(X_1) + \dots + \phi_n(X_n). \quad (5)$$

In Rockafellar (1997) Theorems 5.4 and 16.4 it is shown that  $(\phi_1 \square \dots \square \phi_n)^* = \phi_1^* + \dots + \phi_n^*$ . By using the arguments above one can easily see that if  $\phi_1, \dots, \phi_n$  are positive homogeneous then  $\phi_1 \square \dots \square \phi_n(X) = \sup_{Y \in \cap_i \Delta_{\phi_i}} E(YX)$ . As a result

**Theorem 1.** *The infimum in the infimal convolution is bounded if and only if  $\cap_i \Delta_{\phi_i} \neq \emptyset$ .*

Another classical result is the following

**Theorem 2.** *Assume that  $\phi_1, \dots, \phi_n$  are  $n$  positive homogenous convex function. The following two statements are equivalent*

1.  $(X_1, \dots, X_n)$  is an optimal allocation for  $X$  i.e.,  $X_1 + \dots + X_n = X$  and  $\phi_1(X_1) + \dots + \phi_n(X_n) = \phi_1 \square \dots \square \phi_n(X)$ ;
2. There exists  $Y \in L^q$  such that  $\phi_i(X_i) = E(Y X_i), i = 1, \dots, n$ .

For a proof one can see Jouini et al. (2008). Let  $M_1, \dots, M_n$  are  $n$  convex and closed cones, subsets of  $L^p$ , representing  $n$  constraints that agents 1 to  $n$  face in the economy. Then by replacing  $\phi_i$  with  $\phi^{M_i}$  in the above, we can consider the same setting which also incorporates the economy constraint in the problem.

And finally the positive infimal convolution is denoted by  $\varrho_1 \square \dots \square \varrho_n$  as is defined as

$$\varrho_1 \square \dots \square \varrho_n(X) = \inf_{X_1 + \dots + X_n = X, X_i \geq 0, i=1, \dots, n} \phi_1(X_1) + \dots + \phi_n(X_n).$$

### 3 Problem Set-up

Let us assume there are  $n$  different agents in the market whose preferences are according to  $n$  distortion risk measures  $\varrho_1, \dots, \varrho_n$ . We denote the associated kernels with  $\Phi_1, \dots, \Phi_n$ . The risk of the whole market is modeled by a loss variable  $X_0$ . The set of allocations denoted by  $\mathbb{A}$  is defined as follows

$$\mathbb{A} = \{(X_1, \dots, X_n) \in (L^p)^n \mid X_1 + \dots + X_n = X_0\}.$$

An optimal allocation is an allocation which minimizes the aggregate risk

$$\inf_{X_1 + \dots + X_n = X_0} \varrho_1(X_1) + \dots + \varrho_n(X_n), \quad (6)$$

An asymptotic optimal allocation is a sequence  $\{(X_1^m, \dots, X_n^m)\}_{m=1,2,\dots} \subseteq \mathbb{A}$ , such that

$$\varrho_1(X_1^m) + \dots + \varrho_n(X_n^m) \xrightarrow{m \rightarrow \infty} \inf_{X_1 + \dots + X_n = X_0} \varrho_1(X_1) + \dots + \varrho_n(X_n). \quad (7)$$

It is clear that the existence of an asymptotic optimal allocation is equivalent to the boundedness of (6). For further development of the existing setting we have to consider a wider problem

$$\inf_{(X_1, \dots, X_n) \in \mathbb{A}} \lambda_1 \varrho_1(X_1) + \dots + \lambda_n \varrho_n(X_n), \quad (8)$$

when  $(\lambda_1, \dots, \lambda_n)$  is an arbitrary set of positive numbers. For instance, Pareto allocations in a market whose agent utilities are  $-\varrho_i, i = 1, \dots, n$ , are the solutions to

this problem. We will see that if there is no friction in the market, then for any set of coherent risk measures  $\varrho_1, \dots, \varrho_n$ ,  $\lambda_i$ 's should be equal. On the other hand, in (re-)insurance studies, one can find a risk sharing problem which has very similar components;  $\varrho_1$  is a risk measure, measuring the ceding company global risk, and  $\varrho_2$  is a risk premium function, pricing the reinsurance contracts. In this problem  $\lambda_1 = 1$  and  $\lambda_2 = 1 + \rho$  is a relative safely loading parameter (for more details see example below).

### 3.1 General Solutions

Our approach in this section is to reduce the risk allocation problem to an inner problem which can be solved by the existing results in the literature. Even though the general form of a distortion risk is not a coherent risk measure, thanks to the following statement we can use the convex analysis approaches to study (8).

**Theorem 3.** (*Infimal Characterization of distortion Risks*) *Let*

$$\varrho_\Phi(X) = \int_0^1 \text{VaR}_s(X) d\Phi(s),$$

*for a non-decreasing function  $\Phi$  as in Definition 1. If  $\varrho_\Phi$  is  $L^p$  continuous, and  $X$  is bounded below, we have the following equality*

$$\varrho_\Phi(X) = \min\{\varrho(X) | \text{for all l.s.c. coherent risk measures } \varrho \text{ such that } \varrho \geq \varrho_\Phi\}.$$

*Proof.* First, we prove the theorem for  $p = \infty$ . Consider a sequence of partitions  $\Sigma_k = \{\alpha_0^k = 0 < \alpha_1^k < \dots, \alpha_k^k < \alpha_{k+1}^k = 1\}$ ,  $k = 1, 2, \dots$  of  $[0, 1]$  such that  $\Sigma_k \subseteq \Sigma_{k+1}$  and  $\text{mesh}(\Sigma_k) \rightarrow 0$ . According to Theorem 6.8. in Delbaen (2000), for a given  $X$  there are coherent risk measures  $\varrho_i^k$ ,  $i = 1, \dots, k$  such that  $\varrho_i^k \geq \text{VaR}_{\alpha_i^k}$  and  $\varrho_i^k(X) = \text{VaR}_{\alpha_i^k}(X)$ . Define the following mappings on  $L^\infty$ :

$$V_k(Y) = \sum_{i=0}^k (\Phi(\alpha_{i+1}^k) - \Phi(\alpha_i^k)) \text{VaR}_{\alpha_i^k}(Y) \text{ and } \varrho_k(Y) = \sum_{i=0}^k (\Phi(\alpha_{i+1}^k) - \Phi(\alpha_i^k)) \varrho_i^k(Y).$$

Define the coherent risk measure  $\varrho$  as

$$\varrho(Y) = \limsup_k \varrho_k(Y), \forall Y \in L^\infty.$$

It is clear that  $\varrho$  is  $\sigma(L^\infty, L^1)$ -l.s.c. Since  $\varrho_k \geq V_k$  and  $\varrho_k(X) = V_k(X)$ , by using the very definition of an integral it turns out that  $\varrho \geq \varrho_\Phi$  and  $\varrho(X) = \varrho_\Phi(X)$ .



Now, let us assume that  $X \in L^p$ . Let  $\Sigma$  be the set of all finite sigma algebras on  $\Omega$ . Recall that  $\Sigma$  is a directed set. For any  $\mathcal{G} \in \Sigma$  let  $\varrho_{\mathcal{G}}$  be a  $\sigma(L^\infty, L^1)$ -l.s.c coherent risk measure which dominates  $\varrho_\Phi$  on  $L^\infty$  and  $\varrho_\Phi(E(X|\mathcal{G})) = \varrho_{\mathcal{G}}(E(X|\mathcal{G}))$ . Let  $\varrho(Y) = \limsup_{\mathcal{G}} \varrho_{\mathcal{G}}(E(Y|\mathcal{G}))$ ,  $\forall Y \in L^p$ . Notice that for any  $Z \in L^1$ , if  $Y_k \rightarrow Y$  weakly in  $L^p$ ,  $E(ZE(Y_k|\mathcal{G})) = E(E(Z|\mathcal{G})Y_k) \rightarrow E(E(Z|\mathcal{G})Y) = E(ZE(Y|\mathcal{G}))$ . Hence, each function  $Y \mapsto \varrho_{\mathcal{G}}(E(Y|\mathcal{G}))$  is  $L^p$  lower semicontinuous. This implies that  $\varrho$  is also  $L^p$  lower semicontinuous. On the other hand, for any  $Y \in L^p$ , the net  $\{Y_{\mathcal{G}} = E(Y|\mathcal{G})\}_{\mathcal{G}}$  converges in  $L^p$  and therefore, converges in distribution to  $Y$ . This implies that the sequence of functions  $\{t \mapsto \text{VaR}_t(E(Y|\mathcal{G}))\}_{\mathcal{G}}$  converges point-wise to the function  $t \mapsto \text{VaR}_t(Y)$ . Given that  $X$  is bounded below, by using a version of the Fatou lemma for nets, we have that

$$\begin{aligned} \varrho_\Phi(Y) &= \int_0^1 \text{VaR}_t(Y) d\Phi(t) \leq \liminf_{\mathcal{G}} \int_0^1 \text{VaR}_t(Y_{\mathcal{G}}) d\Phi(t) \\ &= \liminf_{\mathcal{G}} \varrho_\Phi(Y_{\mathcal{G}}) \leq \liminf_{\mathcal{G}} \varrho_{\mathcal{G}}(E(Y|\mathcal{G})) \leq \limsup_{\mathcal{G}} \varrho_{\mathcal{G}}(E(Y|\mathcal{G})) = \varrho(Y) \end{aligned}$$

With a similar argument as above, one can show that if instead of Fatou lemma we use the dominated convergence theorem, and also the assumption that  $\varrho_\Phi$  is  $L^p$  continuous, we have that  $\varrho_\Phi(X) = \varrho(X)$ .  $\square$

The following theorem is almost an immediate result from the previous theorem and Theorem 2.

**Theorem 4.** *Let  $\varrho_1, \dots, \varrho_n$  be  $n$   $L^p$ -continuous distortion risk measures,  $\lambda_1, \dots, \lambda_n$ , be  $n$  positive numbers and  $M_1, \dots, M_n$  be  $n$  closed convex cones. Let us denote by  $\Lambda_i^{M_i}$  the set of all functionals  $\tilde{\varrho}_i^{M_i} = \tilde{\varrho}_i + \chi_{M_i}$ , where  $\tilde{\varrho}_i$  is a coherent risk measure greater than or equal to  $\varrho_i$ ,  $i = 1, \dots, n$ . If  $X_0$  is bounded below, the following statements for an allocation  $(X_1, \dots, X_n) \in \mathbb{A}$  hold*

1. *If  $(X_1, \dots, X_n)$  is an optimal allocation for problem (6) for all  $(\lambda_1 \tilde{\varrho}_1^{M_1}, \dots, \lambda_n \tilde{\varrho}_n^{M_n}) \in \Lambda^{M_1} \times \dots \times \Lambda^{M_n}$ , then it is optimal for  $(\lambda_1 \varrho_1^{M_1}, \dots, \lambda_n \varrho_n^{M_n})$ .*
2. *If  $(X_1, \dots, X_n)$  is not optimal for any  $(\lambda_1 \tilde{\varrho}_1^{M_1}, \dots, \lambda_n \tilde{\varrho}_n^{M_n}) \in \Lambda^{M_1} \times \dots \times \Lambda^{M_n}$ , then it is not optimal for  $(\lambda_1 \varrho_1^{M_1}, \dots, \lambda_n \varrho_n^{M_n})$ .*
3. *If  $(X_1, \dots, X_n)$  is an optimal allocation for  $(\lambda_1 \varrho_1^{M_1}, \dots, \lambda_n \varrho_n^{M_n})$  then there exists  $Y \in L^q$  such that  $\lambda_i \varrho_i^{M_i}(X_i) = E(X_i Y)$ ,  $i = 1, \dots, n$ .*

*Remark 2.* From pricing point of view, in the third statement of the previous theorem,  $Y$  can be interpreted as the “generalized stochastic discount factor”. For further

reading on the relation between the set of stochastic discount factors and the optimal risk allocations see Filipović and Kupper (2008a).

In the following theorem we study the existence of an asymptotic optimal allocation.

**Theorem 5.** *Let  $\varrho_1, \dots, \varrho_n$  be  $n$  distortion risk measures, and for each  $i=1, \dots, n$ , let  $\Lambda_i$  denote the set of all coherent risk measures  $\tilde{\varrho}_i \geq \varrho_i$ . If the total risk  $X_0$  is bounded below by  $M \in \mathbb{R}$ , (8) is bounded if and only if  $\cap_i \lambda_i \Delta_{\tilde{\varrho}_i} \neq \emptyset$  for all  $(\tilde{\varrho}_1, \dots, \tilde{\varrho}_n) \in \Lambda_1 \times \dots \times \Lambda_n$ .*

*Proof.* By Theorem 3

$$\begin{aligned}
& \inf_{X_1 + \dots + X_n = X_0} \lambda_1 \varrho_1(X_1) + \dots + \lambda_n \varrho_n(X_n) \\
&= \inf_{X_1 + \dots + X_n = X_0} \left\{ \min_{\tilde{\varrho}_1 \in \Lambda_1} \lambda_1 \tilde{\varrho}_1(X_1) + \dots + \min_{\tilde{\varrho}_n \in \Lambda_n} \lambda_n \tilde{\varrho}_n(X_n) \right\} \\
&= \inf_{(\tilde{\varrho}_1, \dots, \tilde{\varrho}_n) \in \Lambda_1 \times \dots \times \Lambda_n} \inf_{X_1 + \dots + X_n = X} \left\{ \lambda_1 \tilde{\varrho}_1(X_1) + \dots + \lambda_n \tilde{\varrho}_n(X_n) \right\} \\
&= \inf_{(\tilde{\varrho}_1, \dots, \tilde{\varrho}_n) \in \Lambda_1 \times \dots \times \Lambda_n} \sup_{Y \in \cap_i \lambda_i \Delta_{\tilde{\varrho}_i}} E(Y X_0). \quad (9)
\end{aligned}$$

It is clear that if the infimum in (9) is bounded then all intersections  $\cap_i \lambda_i \Delta_{\tilde{\varrho}_i}$ , for all  $(\tilde{\varrho}_1, \dots, \tilde{\varrho}_n) \in \Lambda_1 \times \dots \times \Lambda_n$ , are non-empty. On the other hand, since  $\forall Y \in \Delta_{\tilde{\varrho}_i}, i = 1, \dots, n, Y \geq 0, E(Y) = 1$  and  $X_0 \geq M$ , we have that  $E(Y X_0) \geq -|M|$ . This implies that if all the intersections  $\cap_i \lambda_i \Delta_{\tilde{\varrho}_i}$ , for all  $(\tilde{\varrho}_1, \dots, \tilde{\varrho}_n) \in \Lambda_1 \times \dots \times \Lambda_n$ , are non-empty, then the right hand side of (9) is bounded below by  $-|M| \max_i \lambda_i$ , and therefore, (9) is bounded.  $\square$

Now we have the following corollaries

**Corollary 1.** *The boundedness of the problem (8) is independent of the total risk.*

**Corollary 2.** *For  $X_0 \geq 0$ , (8) has a solution if and only if  $\lambda_1 = \dots = \lambda_n$  and  $\cap_i \Delta_{\tilde{\varrho}_i} \neq \emptyset$ .*

**Example 1.** Let  $\varrho_1 = \text{VaR}_\alpha$  and  $\varrho_2 = E$ , and let us assume  $X_0$  is any arbitrary random loss. According to Theorem 5, the optimal risk allocation problem (8) has a solution if  $P \in \Delta_{\tilde{\varrho}}$  for any coherent risk measure  $\tilde{\varrho} \geq \text{VaR}_\alpha$ . On the other hand, according to Theorem 3 for any  $X \in L^p$ ,  $\text{VaR}_\alpha(X) = \tilde{\varrho}(X)$  for some coherent risk measure  $\tilde{\varrho} \geq \text{VaR}_\alpha$ . This implies that  $\text{VaR}_\alpha(X) \geq E(X)$ , for any  $X \in L^p$ . This inequality clearly does not hold, if we choose  $X = 1_A$  for some set  $A \in \mathcal{F}$  that  $0 < P(A) < \frac{1-\alpha}{2}$ .

Theorems 4 and 5 can be considered as generalization of many existing papers in the literature where their result can only be applied to coherent risk measures, which in our setting is to use singleton sets  $\Lambda_i = \{\varrho_i\}$ ; see for instance Jouini et al. (2008) , Filipović and Kupper (2008a) and Filipović and Kupper (2008b)

### 3.2 Co-monotone allocations and Admissible Allocations

Concerning the discussion we had about moral hazard, in this section we assume that all contracts in the market are designed to be co-monotone. To set this economic assumption on a sound mathematical basis, we assume that all contracts are non-decreasing functions of the total risk. Therefore, in a market with this assumption any allocation  $(X_1, \dots, X_n)$  is equal to  $(f_1(X_0), \dots, f_n(X_0))$  when  $f_1, \dots, f_n$  are  $n$  non-negative and non-decreasing functions such that  $f_1 + \dots + f_n = id$ .

We introduce the set of allocations as

$$\mathbb{C} = \left\{ f \in L_+^0(\mathbb{R}_+) \mid f \text{ is nondecreasing and } f(0) = 0 \right\}.$$

and the set of admissible allocation as

$$\mathbb{AC} = \{(f_1, \dots, f_n) \in \mathbb{C}^n \mid f_1 + \dots + f_n = id\}.$$

$\mathbb{AC}$  is a closed, convex and weakly compact set of  $L^p(\mathbb{R})$  for  $p \in [1, \infty)$ . On the other hand, it is easy to see that any component  $f_i$ , is a Lipschitz function of degree one, i.e.  $0 \leq f_i(y) - f_i(x) \leq y - x$ , for  $0 \leq x \leq y$ . Indeed, it is enough to check it for  $n = 2$ . In this paper, we focus our attention to the allocation set induced by  $\mathbb{AC}$

$$\mathbb{AA} = \{(f_1(X_0), \dots, f_n(X_0)) \mid (f_1, \dots, f_n) \in \mathbb{AC}\}.$$

Filipović and Svindland (2008) prove that for a set of  $n$  law and cash invariant convex functions  $\varrho_1, \dots, \varrho_n$ , any solution  $(X_1, \dots, X_n)$  to (6) is co-monotone. In particular this means that in a market with convex distortion risks the optimal allocations are automatically from  $\mathbb{AC}$ . This is no longer true for the general case as shown in the following example.

**Example 2.** Let us assume  $\varrho_1 = \text{VaR}_\alpha$ ,  $\varrho_2 = \text{VaR}_\beta$ ,  $X_0 > 0$ , *a.s.*,  $\alpha + \beta > 1$  and  $0 < \alpha < \beta < 1$ . Let us assume  $X_0$  is a random variable with a strictly increasing and continuous CDF function  $F_{X_0}$ . Since  $n = 2$  in this example, one can assume that there is a function  $f$  such that  $f$  and  $id - f$  are non-negative, non-decreasing and that  $f_1 = f$  and  $f_2 = id - f$ . We first prove the following lemma

**Lemma 1.** *There is a positive number  $c > 0$  such that for any function  $f$  described above, the following inequality holds*

$$\text{VaR}_\alpha(f(X_0)) + \text{VaR}_\beta(X_0 - f(X_0)) > c + \text{VaR}_{\alpha+\beta-1}(X_0).$$

*Proof.* It is known that Value at Risk can commute with a non-decreasing function, therefore,

$$\begin{aligned}\text{VaR}_\alpha(f(X_0)) &= f(\text{VaR}_\alpha(X_0)), \\ \text{VaR}_\beta(X_0 - f(X_0)) &= \text{VaR}_\beta(X_0) - f(\text{VaR}_\beta(X_0)).\end{aligned}$$

Strict monotonicity of  $F_{X_0}$ ,  $\alpha < \beta$  and  $\alpha + \beta - 1 < \alpha$  imply

$$\begin{aligned}\text{VaR}_\alpha(f(X_0)) + \text{VaR}_\beta(X_0 - f(X_0)) &= f(\text{VaR}_\alpha(X_0)) + \text{VaR}_\beta(X_0) - f(\text{VaR}_\beta(X_0)) \\ &\geq \text{VaR}_\beta(X_0) + (\text{VaR}_\alpha(X_0) - \text{VaR}_\beta(X_0)) \\ &= \text{VaR}_\alpha(X_0) > c + \text{VaR}_{\alpha+\beta-1}(X_0),\end{aligned}$$

where  $c = \frac{\text{VaR}_\alpha(X_0) - \text{VaR}_{\alpha+\beta-1}(X_0)}{2}$ . □

The result of the lemma is that there is no *admissible allocation* which can attain the value  $\text{VaR}_{\alpha+\beta-1}(X_0)$ . Now let us consider the allocation  $X_1 = X_0 1_{\{X_0 > \text{VaR}_\alpha(X_0)\}}$ . It is clear that  $P(X_1 > 0) = 1 - \alpha$ , meaning that  $\text{VaR}_\alpha(X_1) = 0$ . On the other hand,

$$\begin{aligned}P(X_2 > x) &= P(X_0 > x \ \& \ \text{VaR}_\alpha(X_0) \geq X_0) \\ &= P(X_0 \leq \text{VaR}_\alpha(X_0)) - P(X_0 \leq x) \\ &= \alpha - F_{X_0}(x).\end{aligned}$$

This simply implies that  $F_{X_2}(x) = 1 + F_{X_0}(x) - \alpha$ , and therefore  $\text{VaR}_\beta(X_2) = \text{VaR}_{\alpha+\beta-1}(X_0)$ . Hence, we have that  $\text{VaR}_\alpha(X_1) + \text{VaR}_\beta(X_2) = \text{VaR}_{\alpha+\beta-1}(X_0)$ .

Allocation  $(X_1, X_2)$  is an example of a moral hazard situation, where agent 2 is not sensitive to the big total losses. This example shows why in a market with non-convex beliefs we have to further assume that there is no risk of moral hazard.

*Remark 3.* Observe that if all agents in the market use the same risk measure  $\varrho$ , by using the fact that VaR commutes with non-decreasing functions, we have

$$\varrho(f_1(X_0)) + \dots + \varrho(f_n(X_0)) = \varrho(X_0), \forall (f_1, \dots, f_n) \in \mathbb{AC}.$$

This means, no matter what allocation the agents use, as far as there is no risk of moral hazard, the value of the systemic risk remains constant. This may happen if the regulator imposes a unique risk measure to all agents, for example the same  $\text{VaR}_{0.995}$  as in the Solvency II, to measure the capital reserve.

### 3.3 Marginal Risk Allocations

It is known that every Lipschitz continuous function  $f$  is almost everywhere differentiable and its derivative is essentially bounded by its Lipschitz constant. Furthermore,  $f$  can be written as the integral of its derivative denoted by, i.e.,  $f(x) = \int_0^x h(t)dt$ . Therefore, the set  $\mathbb{C}$  can be represented as

$$\mathbb{C} = \left\{ f \in L^0(\mathbb{R}_+) \mid f(x) = \int_0^x h(t)dt, 0 \leq h \leq 1 \right\}.$$

Let us introduce the space of *marginal risk allocations* as

$$\mathbb{D} = \left\{ h \in L^0(\mathbb{R}_+) \mid 0 \leq h \leq 1 \right\}.$$

**Definition 3.** For any function  $f \in \mathbb{C}$ , the associated marginal risk allocation is a function  $h \in \mathbb{D}$  such that

$$f(x) = \int_0^x h(t)dt, x \geq 0.$$

The interpretation of marginal risk allocation is as follows: if  $f(x) = \int_0^x h(t)dt$  is in  $\mathbb{C}$ , then at each value  $X_0 = x$ , a marginal change  $\delta$  to the value of the total risk will result in marginal change of the size  $\delta h(x)$  in the allocation risk. We will see in the following that this marginal change is either 0 or  $\delta$ , i.e.,  $h = 0$  or 1. This means that for any small change in the total risk, there is only one agent who has to tolerate the changes in the risk.

### 3.4 Co-monotone Optimal Risk Allocations

Throughout this section we assume  $X_0 \geq 0$  and  $F_{X_0}(0) = 0$ . Furthermore, we restrict our attention to a family of distortion risk measures which satisfy the following regularity condition

$$\lim_{m \rightarrow \infty} \varrho_i(X \wedge m) = \varrho_i(X), i = 1, \dots, n. \quad (10)$$

Let  $\Psi(t) = \min \{\lambda_1(1 - \Phi_1(t)), \dots, \lambda_n(1 - \Phi_n(t))\}$ . Suppose  $k_i^*, i = 1, \dots, n$  is a set of functions that

$$k_i^*(t) = \begin{cases} 1, & \text{if } \lambda_i(1 - \Phi_i(t)) < \lambda_j(1 - \Phi_j(t)) \forall i \neq j \\ 0, & \text{if } \lambda_i(1 - \Phi_i(t)) > \lambda_j(1 - \Phi_j(t)) \exists i \neq j \end{cases}, \quad (11)$$

where also  $k_1^* + \dots + k_n^* = 1$ . Here we state the main result of this section

**Theorem 6.** *If  $\varrho_1, \dots, \varrho_n$  satisfy (10), the co-monotone solutions to the optimization problem (8) is given by  $X_i = f_i^*(X_0)$  when*

$$f_i^*(x) = \int_0^x k_i^*(\text{VaR}_t(X_0)) dt, i = 1, \dots, n. \quad (12)$$

Furthermore, the value at minimum is given by

$$\int_0^\infty \Psi(s) ds. \quad (13)$$

*Proof.* Let  $\varrho_i = \int_0^1 \text{VaR}_t(X_0) d\Phi_i(t)$ ,  $i = 1, \dots, n$ . Then for any member  $(f_1, \dots, f_n)$  from the set  $\mathbb{AC}$ , using the fact that VaR always commutes with non-decreasing functions, we have

$$\begin{aligned} & \lambda_1 \varrho_1(f_1(X_0)) + \dots + \lambda_n \varrho_n(f_n(X_0)) \\ &= \int_0^1 \lambda_1 \text{VaR}_t(f_1(X_0)) d\Phi_1(t) + \dots + \int_0^1 \lambda_n \text{VaR}_t(f_n(X_0)) d\Phi_n(t) \\ &= \int_0^1 \lambda_1 f_1(\text{VaR}_t(X_0)) d\Phi_1(t) + \dots + \int_0^1 \lambda_n f_n(\text{VaR}_t(X_0)) d\Phi_n(t). \end{aligned} \quad (14)$$

Let us denote the derivatives of  $f_1, \dots, f_n$  by  $h_1, \dots, h_n$ . Therefore,

$$\begin{aligned} & \lambda_1 \varrho_1(f_1(X_0)) + \dots + \lambda_n \varrho_n(f_n(X_0)) \\ &= \int_0^1 \left( \int_0^{\text{VaR}_t(X_0)} \lambda_1 h_1(s) ds \right) d\Phi_1(t) + \dots + \int_0^1 \left( \int_0^{\text{VaR}_t(X_0)} \lambda_n h_n(s) ds \right) d\Phi_n(t). \end{aligned}$$

First, we assume  $X_0$  is bounded. By Fubini's Theorem we have

$$\begin{aligned} & \lambda_1 \varrho_1(f_1(X_0)) + \dots + \lambda_n \varrho_n(f_n(X_0)) \\ &= \int_0^\infty \left[ \left( \int_{F_{X_0}(s)}^1 \lambda_1 d\Phi_1(t) \right) h_1(s) + \dots + \left( \int_{F_{X_0}(s)}^1 \lambda_n d\Phi_n(t) \right) h_n(s) \right] ds \\ &= \int_0^\infty [\lambda_1 (1 - \Phi_1(F_{X_0}(s))) h_1(s) + \dots + \lambda_n (1 - \Phi_n(F_{X_0}(s))) h_n(s)] ds \end{aligned} \quad (15)$$

where we use the fact that  $\Phi_1(1) = \dots = \Phi_n(1) = 1$ . It is now clear that the following  $(h_1^*, \dots, h_n^*)$  will minimize (15)

$$h_i^*(s) = \begin{cases} 1, & \text{if } \lambda_i(1 - \Phi_i(F_{X_0}(s))) < \lambda_j(1 - \Phi_j(F_{X_0}(s))), \forall i \neq j \\ 0 & \text{if } \lambda_i(1 - \Phi_i(F_{X_0}(s))) > \lambda_j(1 - \Phi_j(F_{X_0}(s))), \exists i \neq j \end{cases} \quad (16)$$

where also  $h_1^* + \dots + h_n^* = 1$ . The value of the minimum also is equal to

$$\int_0^\infty \Psi(s) ds. \quad (17)$$

If we make a simple change of variable  $t = F_{X_0}(s)$ , we get the result.

Now assume the general case when  $X_0$  is not bounded. It is clear that at each point  $t$ , for every  $i$  between 1 and  $n$ ,  $\{\Phi_i \circ F_{X_0 \wedge m}(t)\}_{m=1}^\infty$  is non-increasing with respect to  $m$ . On the other hand, for any  $t$ , there exist  $m_t$  such that if  $m > m_t$  then  $F_{X_0 \wedge m}(t) = F_{X_0}(t)$ . Therefore, at each point  $t$ , we have that  $\Phi_i(F_{X_0 \wedge m}(t)) \downarrow \Phi_i(F_{X_0}(t))$ . By monotone convergence theorem we have that

$$\lim_{m \rightarrow \infty} \int_0^\infty \Phi_i(F_{X_0 \wedge m}(t)) h(t) dt = \int_0^\infty \Phi_i(F_{X_0}(t)) h(t) dt,$$

for any function  $h \in \mathbb{D}$ . Using this fact and our continuity assumption

$$\begin{aligned} \varrho_i(f(X_0)) &= \lim_{m \rightarrow \infty} \varrho_i(f(X_0) \wedge f(m)) \\ &= \lim_{m \rightarrow \infty} \varrho_i(f(X_0 \wedge m)) \\ &= \lim_{m \rightarrow \infty} \int_0^\infty (1 - \Phi_i(F_{X_0 \wedge m}(s))) h(s) ds \\ &= \int_0^\infty (1 - \Phi_i(F_{X_0}(s))) h(s) ds \end{aligned}$$

This simply results in

$$\begin{aligned} &\lambda_1 \varrho_1(f_1(X_0)) + \dots + \lambda_n \varrho_n(f_n(X_0)) \\ &= \int_0^\infty [\lambda_1(1 - \Phi_1(F_{X_0}(s))) h_1(s) + \dots + \lambda_n(1 - \Phi_n(F_{X_0}(s))) h_n(s)] ds \end{aligned}$$

The rest of the proof follows the same lines after (15).  $\square$

*Remark 4.* As one can see from the last theorem,  $k_i^*, i = 1, \dots, n$  only depend on market preferences, and therefore, they are universal. Also one can see from (12) how the role of the total risk and the market preferences are separated.

*Remark 5.* In Theorem 6, it is shown that the marginal risk allocations take only the values zero or one. There are some similar results in the literature of actuarial mathematics which can prove this for very particular settings, for instance; see Cai et al. (2008), Cheung (2010), Chi (2012b), Chi (2012a), Chi and Tan (2013), Cheung et al. (2014) and more recently Assa (2015). Theorem 6 can extend all those works from two different aspects. First, we use a larger family of risk measures and premiums (distortion risk measures and premiums) which include almost all risk measures such as VaR and CVaR and risk premiums such as Wang's premium, used by them. Second, our work can increase the number of players from two (insurance and re-insurance company) to  $n$ , which otherwise, by using the techniques from the existing literature would be either impossible or at least very difficult to do.

**Corollary 3.** *If  $\lambda_1 = \dots = \lambda_n$  then  $\varrho_1 \square \dots \square \varrho_n = \varrho_\Phi$  when  $\Phi = \max \{\Phi_1, \dots, \Phi_n\}$ .*

**Example 3.** Let us consider the example we discussed earlier. Let us consider that there are two companies using  $\varrho_1 = \text{VaR}_\alpha$  and  $\varrho_2 = \text{VaR}_\beta$ , where  $\alpha < \beta$ . It is clear that since  $\alpha < \beta$  one solution is  $h_1 = 1$  and  $h_2 = 0$  and  $\varrho_1 \square \varrho_2(X_0) = \text{VaR}_\alpha(X_0)$ .

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